BRUSHING THE HAIRS
OF TRANSCENDENTAL ENTIRE FUNCTIONS

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ABSTRACT. Let f be a transcendental entire function of finite order in the Eremenko-Lyubich class $\mathcal{B}$ (or a finite composition of such maps), and suppose that f is hyperbolic and has a unique Fatou component. We show that the Julia set of f is a *Cantor bouquet*; i.e. is ambiently homeomorphic to a straight brush in the sense of Aarts and Oversteegen. In particular, we show that any two such Julia sets are ambiently homeomorphic.

We also show that if $f \in \mathcal{B}$ has finite order (or is a finite composition of such maps), but is not necessarily hyperbolic with connected Fatou set, then the Julia set of f *contains* a Cantor bouquet.

As part of our proof, we describe, for an arbitrary function $f \in \mathcal{B}$, a natural compactification of the dynamical plane by adding a "circle of addresses" at infinity.

1. INTRODUCTION

In recent decades there has been an increasing interest in studying the dynamics generated by the iterates of transcendental entire functions. For these dynamical systems, the presence of an essential singularity at infinity creates some major differences to the dynamics of polynomials or rational maps.

More precisely, let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. Then the *Julia set* $J(f)$ is the set of points $z \in \mathbb{C}$ at which the family $\{f^n\}_{n>0}$ is not equicontinuous with respect to the spherical metric (this is where the dynamics of f is "chaotic"). The complement $F(f) = \mathbb{C} \setminus J(f)$—i.e. the set of stable behaviour—is called the *Fatou set*. Due to the essential singularity at infinity, the Julia sets of transcendental entire functions tend to be far more complicated than for rational maps, and one of the main problems in the field is to understand these sets (and the dynamics thereon) from a topological and geometric point of view.

A cornerstone example of transcendental entire maps is given by the complex exponential family $E_\lambda(z) = \lambda \exp(z)$, $\lambda \in \mathbb{C}$. It is often considered the simplest possible family of transcendental entire functions, and has received considerable attention since at least the 1980s. For real parameter values $\lambda \in (0, 1/e)$, the Fatou set has a unique connected component, consisting of all points that converge to an attracting fixed point $z_0^\lambda \in \mathbb{R}$ under iteration. In [10] it was proved that for these parameters, the Julia set

is given by a union of pairwise disjoint arcs to ∞, which are called *hairs* or *dynamic rays*. Each point z on each hair, except possibly the finite endpoint of the hair, satisfies $\operatorname{Re}\left(E_\lambda^j\right)(z) \to +\infty$ when $j \to \infty$. Due to its appearance, the union of these hairs has been referred to as a *Cantor bouquet* (a term to which, for the purposes of this article, we will give a precise mathematical meaning below). For a long time, hairs as above have also been known to exist for many other functions, such as the sine family, $z \mapsto \lambda \sin(z)$; see [11].

Aarts and Oversteegen [1] gave a complete topological description of the Julia sets $J(E_\lambda)$, $\lambda \in (0, 1/e)$, as well as the Julia sets $J(\lambda \sin)$ for $\lambda \in (0, 1)$, by showing that all these sets are homeomorphic to a single universal topological object. They did so by explicitly constructing a homeomorphism between the Julia set and a type of subset of $\mathbb{R}^2$ that is called a *straight brush* (see Definition 2.1), and proved that any two straight brushes are homeomorphic. In fact, they even showed that the sets are *ambiently* homeomorphic; i.e., the homemorphism between them can be chosen to extend to a homeomorphism of $\mathbb{R}^2$. (Compare also [7, 20] for a discussion of the topological dynamics of other parameter values in the exponential family.) As mentioned above, we shall refer to sets such as these as *Cantor bouquets*:

Definition 1.1 (Cantor bouquet)**.** A *Cantor bouquet* is any subset of the plane that is ambiently homeomorphic to a straight brush.

Remark. This is different from the terminology in [11]. There a "Cantor n-bouquet" is a certain type of subset of the Julia set that is homeomorphic to the product of a Cantor set with the interval $[0, \infty)$, and a Cantor bouquet is simply the closure of an increasing sequence of Cantor n-bouquets, with $n \to \infty$.

We note that the Cantor bouquets we construct can also be seen to be Cantor bouquets in the sense of Devaney and Tangerman, and that the functions considered in [11] satisfy the hypotheses of our theorems.

Recently, there has been significant progress in extending the above-mentioned description of exponential Julia sets (and those of other explicit maps) to much more general classes of transcendental entire functions. To state these results, we use the following standard definitions.

Definition 1.2 (Hyperbolicity)**.** A transcendental entire function f is called *hyperbolic* if the *postsingular set*

$$\mathcal{P}(f) := \overline{\bigcup_{j \geq 0} f^j(\operatorname{Sing}(f^{-1}))}$$

is a compact subset of the Fatou set $F(f)$.

The function f is called *of disjoint type* if it is hyperbolic and furthermore $F(f)$ is connected. (Equivalently, $\overline{\operatorname{Sing}(f^{-1})}$ is a compact subset of the immediate attracting basin of an attracting fixed point.)

Remark. Here $\operatorname{Sing}(f^{-1})$ denotes the set of *singularities of f^{-1}*. $\operatorname{Sing}(f^{-1})$ consists of the critical and asymptotic values of f. Recall that an *asymptotic value* of f is a point $z_0 \in \mathbb{C}$ for which there is a curve $\alpha(t) : [0, \infty) \to \mathbb{C}$ satisfying $|\alpha(t)| \to \infty$ and $f(\alpha(t)) \to z_0$ as $t \to \infty$. For example, 0 is the unique singular value of the exponential map E_λ.

Note that our definition of hyperbolicity implies, in particular, that the set $\operatorname{Sing}(f^{-1})$ is bounded. The set of all transcendental entire functions with the latter property is known as the *Eremenko-Lyubich class* and denoted by $\mathcal{B}$.

Definition 1.3 (Finite order). A transcendental entire function has *finite order* if there are constants $c, \rho > 0$ such that $|f(z)| \leq c \cdot e^{|z|^\rho}$ for all $z \in \mathbb{C}$.

The following result was established in [3, 26]:

Theorem 1.4 ([3, Theorem C], [26, Theorem 5.10]). *Let f be a disjoint-type function of finite order. Then $J(f)$ is a union of pairwise disjoint arcs to infinity.*

The conclusion holds, more generally, if f is a disjoint-type function that can be written as a finite composition of finite-order functions in the class $\mathcal{B}$.

It was also claimed without proof in [23, 26] that, under the hypotheses of the theorem, the Julia sets are Cantor bouquets (i.e., ambiently homeomorphic to a straight brush in the sense of [1]). In this note, we justify this claim.

Theorem 1.5. *Under the hypotheses of Theorem 1.4, the Julia set $J(f)$ is a Cantor bouquet.*

We note that, as with the results in [26], the theorem holds for a more general class of functions than stated above (see Definition 4.1 and Corollary 6.3). However, it is necessary to make some function-theoretic assumptions in addition from the dynamical requirement of the function being of disjoint type. Indeed, in [26, Theorem 8.4], it is shown that there exists a disjoint-type function f such that $J(f)$ contains no nontrivial curves. So in this case $J(f)$ is very far from being a Cantor bouquet!

Theorem 1.4 and Corollary 6.3 imply, in particular, that all the Julia sets covered by these results are ambiently homeomorphic. We note that this is in contrast to the fact—already noted in [1]—that the geometry and dynamics of Cantor bouquets can be rather diverse. We mention a few examples:

- For hyperbolic exponential maps, the Julia set has zero Lebesgue measure, but for maps in the sine family, the area is positive [16].
- If f is of finite order, then the Julia set always has Hausdorff dimension 2, while the dimension of the set of hairs (without endpoints) is equal to 1 [4]. On the other hand, Stallard [31] showed that the Hausdorff dimension of $J(f)$ can take any value $d \in (1, 2]$, and it can be checked that these examples satisfy the hypotheses of Corollary 6.3.
- For hyperbolic exponential maps, the set of escaping points has Hausdorff dimension 2, and the set of non-escaping points has Hausdorff dimension strictly less than 2 [32]. However, there are examples of finite-order disjoint-type functions for which the set of nonescaping points also has full Hausdorff dimension [24] and functions (of infinite order) for which the escaping set has Hausdorff dimension equal to 1, and the Julia set has dimension arbitrarily close to 1 [27].

It has been known for a long time that hairs exist in the dynamical plane of exponential maps also when the map is not hyperbolic. Indeed, in recent years this approach has been instrumental in obtaining a much deeper understanding of the dynamics for arbitrary parameter values of the exponential family (compare [20, 29, 30] and references therein).

The results in [26] actually establish the existence of hairs for all (not necessarily disjoint type or hyperbolic) maps $f \in \mathcal{B}$ of finite order, as well as their compositions. Essentially, it is proved that the set of all points that stay near infinity under the iteration of such a function consists of hairs. (See Corollary 4.4 below for a precise statement.) We prove a version of our theorem also in this case, which shows in particular that the Julia set of such a function always contains a Cantor bouquet.

Theorem 1.6 (Absorbing Cantor bouquets)**.** *Let f be a finite order function in the class $\mathcal{B}$ (or, more generally, a composition of such functions). Then for every $R > 0$, there exists a Cantor bouquet $X \subset J(f)$ with $f(X) \subset X$ such that*

(1) *$|f^j(z)| \geq R$ for all $z \in X$ and $j \geq 0$;*
(2) *there is $R' \geq R$ such that, if $z \in \mathbb{C}$ with $|f^j(z)| \geq R'$ for all j, then $z \in X$.*

In our proof we use the results from [3, 26], together with the topological characterisation of Cantor bouquets given in [1]. To apply the latter, we compactify the Julia set $J(f)$ by adding a circle (of "addresses") at infinity, with different hairs of $J(f)$ ending at different points of this circle. This construction is, in fact, completely general: If $f \in \mathcal{B}$ is arbitrary, then we describe a natural combinatorial compactification of the dynamical plane, which is likely to be useful in future applications. (Again, this generalizes a construction well-known in the setting of exponential maps.)

Organization of the paper. In Section 2 we recall the topological definitions and results from [1]. In Section 3, we shortly review the *logarithmic change of coordinates*, one of the main tools for studying the class $\mathcal{B}$, and the combinatorial notion of external addresses. Section 4 contains a description of the hairs from Theorem 1.4, while Section 5 constructs a suitable compactification of the Julia set. Finally, we prove our main theorems in Sections 6 and 7.

Basic notation. As usual, we denote the complex plane by $\mathbb{C}$ and the Riemann sphere by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Closures and boundaries will be considered in $\mathbb{C}$ unless explicitly stated otherwise.
   We use the notation $\mathbb{N}_0$ for the set of nonnegative integers.

2. Straight brushes and hairy arcs

In this section, we review some topological notions and results from [1]. We begin with the formal definition of a straight brush.

Definition 2.1 (Straight brush)**.** A subset B of $[0, +\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a *straight brush* if the following properties are satisfied:

- The set B is a closed subset of $\mathbb{R}^2$.
- For every $(x, y) \in B$ there exists $t_y \geq 0$ such that $\{x : (x, y) \in B\} = [t_y, +\infty)$. The set $[t_y, +\infty) \times \{y\}$ is called the hair attached at y and the point (t_y, y) is called its endpoint.
- The set $\{y : (x, y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$ there exist two sequences of hairs attached respectively at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n$, $\beta_n, \gamma_n \to y$ and $t_{\beta_n}, t_{\gamma_n} \to t_y$ as $n \to \infty$.

A simple topological characterization of straight brushes can be found in [8]; in particular, any two such spaces are homeomorphic. In fact, as shown in [1], they are even *ambiently* homeomorphic, which essentially requires that the "vertical" order of the hairs is preserved under the homeomorphism. To discuss this, it is useful to consider a compactification of the brush instead. Hence we make the following definitions, following [1].

Definition 2.2 (Comb). A comb is a continuum (i.e., a compact connected metric space) X containing an arc B (called the base of X) such that:

(1) The closure of every connected component of $X \setminus B$ is an arc, with exactly one endpoint in B. In particular, for every $x \in X \setminus B$, there exists a unique arc $\gamma_x : [0,1] \to X$ with $\gamma_x(0) = x$, $\gamma_x(t) \notin B$ for $t < 1$ and $\gamma_x(1) \in B$. In this case, we say that x belongs to the hair attached at $\gamma_x(1)$.
(2) Distinct components of $X \setminus B$ have disjoint closures.
(3) The set $X \setminus B$ is dense in X.
(4) If $x_0 \in X \setminus B$ and $x_n \in X \setminus B$ is a sequence of points converging to x_0, then $\gamma_{x_n} \to \gamma_{x_0}$ in the Hausdorff metric as $n \to \infty$.

Remark. This definition is formally somewhat different from the one given in [1], but it is not difficult to see that they are equivalent. We decided to use this, rather explicit, form to avoid having to introduce additional topological concepts that are not relevant to the remainder of the paper.

Definition 2.3. A *hairy arc* is a comb with base B and a total order $\prec$ on B (generating the topology of B) such that the following holds. If $b \in B$ and x belongs to the hair attached at b, then there exist sequences x_n^+, x_n^-, attached respectively at points $b_n^+, b_n^- \in B$, such that $b_n^- \prec b \prec b_n^+$ and $x_n^-, x_n^+ \to x$ as $n \to \infty$.

A set $A \subset \mathbb{R}^2$ is called a *one-sided hairy arc* if A is topologically a hairy arc and all hairs are attached to the same side of the base.

Remark 1. Here it should be intuitively clear what "being attached to the same side of the base" means (this is the same terminology as used in [1]). Formally, we can define this notion as follows: there exists a Jordan arc $C \in \mathbb{R}^2 \setminus A$ connecting the two endpoints of the base B such that all connected components of $A \setminus B$ belong to the same connected component of $\mathbb{R}^2 \setminus (B \cup C)$.

Remark 2. In [1], a hairy arc is defined, instead, as a space homeomorphic to what is called a *straight one-sided hairy arc* (in analogy with the notion of a straight brush). It is then shown in [1, Theorem 3.11] that this coincides with the topological definition above.

It is easy to see that, to any straight brush, we can add a base $B = \{(\infty, y) : y \in [-\infty, +\infty]\}$ to obtain a hairy arc. Conversely, for any hairy arc X, the set $X \setminus B$ is homeomorphic to a straight brush; this is implied by the following theorem.

Theorem 2.4 ([1, Theorems 3.2, 3.11 and 4.1]). *Any two hairy arcs are homeomorphic.*
Furthermore, any two one-sided hairy arcs $X_1, X_2 \subset \mathbb{R}^2$ are ambiently homeomorphic. More precisely, any homeomorphism $\varphi : X_1 \to X_2$ extends to a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$.

In particular, any two straight brushes are ambiently homeomorphic.

3. Logarithmic coordinates and the class $\mathcal{B}^{\mathrm{p}}_{\log}$

A useful tool for studying functions in the Eremenko-Lyubich class $\mathcal{B}$ is the *logarithmic change of coordinates*. Let $f \in \mathcal{B}$ and let $D \subset \mathbb{C}$ be a Jordan domain that contains the set $\overline{\mathrm{Sing}(f^{-1})}$ as well as the values 0 and $f(0)$. Set $W := \mathbb{C} \setminus \overline{D}$. The connected components of $\mathcal{V} := f^{-1}(W)$ are called the *tracts* of f. Each tract T is a Jordan domain whose boundary passes through ∞, and $f : T \to W$ is a universal covering. If f has finite order, then the number of tracts will be finite, but otherwise there may be infinitely many tracts (as is the case e.g. for $z \mapsto e^{e^z}$).

It is convenient to study f in logarithmic coordinates. To do so, we set $H := \exp^{-1}(W)$ and $\mathcal{T} := \exp^{-1}(\mathcal{V})$. Each component of $\mathcal{T}$ is a simply connected domain whose boundary is homeomorphic to $\mathbb{R}$ such that both 'ends' of the boundaries have real parts converging to $+\infty$. Note that $\mathcal{T}$ and H are invariant under translation by $2\pi i$. We can lift f to a map
$$F : \mathcal{T} \to H,$$
satisfying $\exp \circ F = f \circ \exp$. We may also assume that the function F is chosen to be $2\pi i$-periodic. We call F a *logarithmic transform* of f. By construction, the following properties hold.

(1) H is a $2\pi i$-periodic unbounded Jordan domain that contains a right half-plane.
(2) $\mathcal{T} \neq \emptyset$ is $2\pi i$-periodic and $\mathrm{Re}\, z$ is bounded from below in $\mathcal{T}$.
(3) Each component T of $\mathcal{T}$ is an unbounded Jordan domain that is disjoint from all its $2\pi i \mathbb{Z}$-translates. For each such T, the restriction $F : T \to H$ is a conformal isomorphism whose continuous extension to the closure of T in $\hat{\mathbb{C}}$ satisfies $F(\infty) = \infty$. (T is called a *tract of F*; we denote the inverse of $F|_T$ by F_T^{-1}.)
(4) The components of $\mathcal{T}$ have pairwise disjoint closures and accumulate only at ∞; i.e., if $z_n \in \mathcal{T}$ is a sequence of points all belonging to different components of $\mathcal{T}$, then $z_n \to \infty$.
(5) F is $2\pi i$-periodic.

Following [26], let us denote by $\mathcal{B}^{\mathrm{p}}_{\log}$ the class of all functions
$$F : \mathcal{T} \to H,$$
where F, $\mathcal{T}$ and H have the properties (1) to (5), regardless of whether F arises as the logarithmic transform of a function $f \in \mathcal{B}$ or not. The advantage of formulating our results in this more general setting is that they can also be applied e.g. to meromorphic functions having logarithmic singularities over infinity, while the proofs remain unchanged.

Remark. To avoid confusion, we should note that, in [23], the notation $\mathcal{B}_{\log}$ is used for the class we denote by $\mathcal{B}^{\mathrm{p}}_{\log}$, while in [26] and other subsequent papers, $\mathcal{B}_{\log}$ denotes the class of functions satisfying (1) to (4), but not necessarily (5). While many of the results[1] we state for the class $\mathcal{B}^{\mathrm{p}}_{\log}$ also hold in this more general setting, our main theorems do not. (The condition of periodicity in (5) could, however, be considerably weakened, at the expense of a more technical definition.)

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[1]The exceptions are Proposition 3.5 and those facts that depend on it, namely Corollary 6.3 and the results of Section 7.

For $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, we define

$$J(F) := \{z \in \overline{\mathcal{T}} : F^j(z) \in \overline{\mathcal{T}} \text{ for all } j \geq 0\}$$

Note that the continuous extension of $F|_T$ in item (3) exists by Carathéodorys Theorem (see [9, Theorem 2.1]). Because T is a Jordan domain, this extension is a homeomorphism, and in particular $F|_T$ extends continuously to a homeomorphism between the closures $\overline{T}$ and $\overline{H}$ (taken in $\mathbb{C}$). Together with property (4), this implies that F extends continuously to the closure $\overline{\mathcal{T}}$ of $\mathcal{T}$ in $\mathbb{C}$. In analogy with our notation for entire functions in the class $\mathcal{B}$, we say that a function $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ is of *disjoint-type* if the boundaries of the tracts of F do not intersect the boundary of H; i.e. if $\overline{\mathcal{T}} \subset H$. The following properties of disjoint type functions are elementary and well-known.

Lemma 3.1 (Properties of disjoint-type functions). *If $F : \mathcal{T} \to H$ is of disjoint type, then F is uniformly expanding with respect to the hyperbolic metric in H. That is, there is a constant $C > 1$ such that the derivative $\|DF(z)\|$ of F with respect to the hyperbolic metric satisfies $\|DF(z)\| \geq C$ for all $z \in \overline{\mathcal{T}}$.*

If $f \in \mathcal{B}$, then the following are equivalent:

(1) *f is of disjoint type;*
(2) *$\mathrm{Sing}(f^{-1})$ is compactly contained in the immediate basin of attraction of an attracting fixed point of f;*
(3) *f has a logarithmic transform F of disjoint type (in which case we have $J(f) = \exp(J(F))$);*

Proof. The first claim is a consequence of the expansion estimate for functions in the class $\mathcal{B}_{\log}^{\mathrm{p}}$ (see Lemma 7.2 below); compare [3, Lemma 3] or [26, Lemma 2.1] for details.

For a proof of the second part, see [18, Proposition 2.7]. $\square$

External addresses. The (Markov-type) partition of the domain of a function $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ by the tracts of F allows us to define symbolic dynamics.

Definition 3.2. Given a function $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, we denote by $\mathcal{A}$ the set of tracts of F. This set $\mathcal{A}$ is called the *symbolic alphabet* associated to F.

We note that, for practical purposes and also in other applications, it is usually more convenient and appropriate to use a symbolic set as address entries, rather than the set of tracts of F itself. For example, suppose that F is the logarithmic transform of a function $f \in \mathcal{B}$ which has finitely many, say N, tracts. Then $\mathcal{A}$ can be identified with the set of pairs (s, j), where $s \in \mathbb{Z}$ and $j \in \{1, \ldots, N\}$. Here j identifies one of the tracts of f, and s determines one of the countably many logarithmic lifts of this domain. However, for our purposes it seems more natural and elegant to use the definition given above.

Definition 3.3 (External addresses). Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, and let $z \in J(F)$. For each $j \geq 0$, let $T_j \in \mathcal{A}$ be the (unique) tract of F with $F^j(z) \in \overline{T_j}$. Then the sequence $\mathrm{addr}(z) := T_0 T_1 T_2 \ldots$ is called the *external address* of z.

Furthermore, we will refer to any sequence $\underline{s} = T_0 T_1 T_2 \cdots \in \mathcal{A}^{\mathbb{N}_0}$ as an *(infinite) external address (for F)*, regardless of whether it is realized as the address of a point

$z \in J(F)$ or not. For such an address $\underline{s}$, we denote the set of points $z \in J(F)$ with $\mathrm{addr}(z) = \underline{s}$ by

$$J_{\underline{s}}(F) := \{z \in J(F) : F^j(z) \in \overline{T_j} \text{ for all } j \geq 0\}.$$

Remark. External addresses are also referred to as *itineraries*. We prefer not to use this terminology, on the one hand to stress the analogy with *external angles* from polynomial dynamics, and on the other in order to not cause confusion with other types of itineraries (defined with respect to certain dynamical partitions).

The alphabet $\mathcal{A}$ is equipped with a natural order, corresponding to the *vertical order* (see [25, p.325]) of the tracts of F near infinity. This gives rise to the *lexicographic order* on the set $\mathcal{A}^{\mathbb{N}_0}$ of external addresses, which we also denote by "$<$".

If $f \in \mathcal{B}_{\log}^{\mathrm{p}}$ and F is a logarithmic transform of f, then we can also define *external addresses for* f by identifying any two addresses $\underline{s} = T_0 T_1 T_2 \dots$ and $\underline{s}' = T_0' T_1' T_2' \dots$ of F with $T_0' = T_0 + 2\pi i k$ for some $k \in \mathbb{Z}$ and $T_j = T_j'$ for $j > 0$. Note that this space is no longer linearly ordered, but carries a circular ordering. For a more natural (but equivalent) way of defining external addresses of f, without using logarithmic coordinates, see e.g. [17, Section 2.3].

The following fact implies that there is a dense set of addresses for which $J_{\underline{s}}(F)$ is nonempty.

Lemma 3.4 ([23, Proposition 2.4]). *Suppose that $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ is a periodic address (i.e., $s_{n+k} = s_n$ for some $k > 0$ and all $n \geq 0$). Then $J_{\underline{s}}(F) \neq \emptyset$.*

Remark. Proposition 2.4 in [23] is stated for fixed addresses; the above statement follows by considering the iterate F^k. It is not difficult to see that the result is also true when the address $\underline{s}$ contains only finitely many symbols.

Accumulation. Recall that a key property in the definition of a hairy arc is that no hair is isolated either from above or from below. For disjoint-type functions, this will be immediate from the following simple fact.

Proposition 3.5 (Accumulation from above and below). *Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ be of disjoint type, and let $z_0 \in J(F)$. Then there exist sequences $z_n^-, z_n^+ \in J(F)$ with $\mathrm{addr}(z_n^-) < \mathrm{addr}(z_0) < \mathrm{addr}(z_n^+)$ for all n and $z_n^- \to z_0$, $z_n^+ \to z_0$.*

Proof. Let $n \geq 1$, and let $\varphi : H \to H$ be the branch of F^{-n} that maps $F^n(z_0)$ to z_0. We define

$$z_n^\pm := \varphi(F^n(z_0) \pm 2\pi i).$$

Then, by definition, we have $\mathrm{addr}(z_n^-) < \mathrm{addr}(z_0) < \mathrm{addr}(z_n^+)$ for all n. Since H contains a right half plane, and because H and $J(F)$ are $2\pi i$-periodic, the hyperbolic distance $\mathrm{dist}_H(z, z + 2\pi i)$ is uniformly bounded independently of $z \in J(F)$. Since F is a strict hyperbolic expansion, it follows that $z_n^\pm \to z_0$ as $n \to \infty$, as claimed. $\square$

4. Existence of hairs: Head-start conditions

Since our proof of the main theorem will strongly use the characterization of hairs from the proof of Theorem 1.4, we shall review some of the definitions and concepts here. This will also allow us to state our results in a more general form, using the notions from [26].

The essential idea of the proof is the following. Suppose F is the logarithmic transform of a function of finite order. Then for any address $\underline{s}$, the set $J_{\underline{s}}(F)$ is naturally ordered according to "escape speed": if $z, w \in J_{\underline{s}}(F)$ are such that $\operatorname{Re} w$ is significantly larger than $\operatorname{Re} z$, then the same will be true for $F(w)$ and $F(z)$ (w has a "head start" over z). This idea is formalized in the following definition.

Definition 4.1 (Head-Start Condition). Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a (not necessarily strictly) increasing continuous function with $\varphi(x) > x$ for all $x \in \mathbb{R}$. We say that F satisfies the *uniform head-start condition for φ* if:

(1) For all tracts T and T' of F and all $z, w \in \overline{T}$ with $F(z), F(w) \in \overline{T'}$,

$$\operatorname{Re} w > \varphi(\operatorname{Re} z) \implies \operatorname{Re} F(w) > \varphi(\operatorname{Re} F(z)) .$$

(2) For all external addresses $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ and for all distinct $z, w \in J_{\underline{s}}$, there is $M \in \mathbb{N}_0$ such that $\operatorname{Re} F^M(z) > \varphi(\operatorname{Re} F^M(w))$ or $\operatorname{Re} F^M(w) > \varphi(\operatorname{Re} F^M(z))$.

The above-mentioned fact for finite-order functions can now be phrased as follows.

Proposition 4.2 ([3, Lemmas 1 and 4] or [26, Section 5]). *Let f belong to the class $\mathcal{B}$ and have finite order, or more generally be a finite composition of such functions. Then there are a logarithmic transform $F : \mathcal{T} \to H$ for f and constants $M > 1$, $K > 0$ such that F satisfies the uniform head-start condition for $\varphi(x) = M \cdot x + K$.*
If f is of disjoint type, then F can also be chosen of disjoint type.

As mentioned above, the proof of Theorem 1.4 relies on the idea of a "speed ordering" generated by the head-start condition. More precisely:

Proposition 4.3 ([26, Propositions 4.4 and 4.6]). *Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ satisfy a uniform head-start condition, say for φ, and let $\underline{s}$ be an external address. We define a relation $\prec$ on $J_{\underline{s}}(F) \cup \{\infty\}$—the "speed ordering"—by*

$$z \prec w \quad \Longleftrightarrow \quad \exists j \geq 0 : F^j(w) > \varphi(F^j(z))$$

and $z \prec \infty$ for all $z \in J_{\underline{s}}(F)$.
Then $\prec$ is a total order, and the order topology coincides with the usual topology on $J_{\underline{s}}(F) \cup \{\infty\}$. In particular, every connected component of $J_{\underline{s}}(F) \cup \{\infty\}$ is an arc.
Moreover, if $J_{\underline{s}}(F) \neq \emptyset$, then $J_{\underline{s}}(F)$ has a unique unbounded component $J_{\underline{s}}^{\infty}(F)$, and for every $z \in J_{\underline{s}}^{\infty}(F)$, the set $\{w \in J_{\underline{s}}(F) \cup \{\infty\} : z \prec w\}$ is an arc connecting z to infinity.

Note that the above result immediately implies Theorem 1.4, since $J_{\underline{s}}(F)$ is connected when F is of disjoint type. To cover also the case where F is not of disjoint type, let us state the following result, which is a consequence of the preceding proposition.

Corollary 4.4 (Hairy Absorbing Sets [26, Theorem 4.7]). *Suppose that $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ satisfies a uniform head-start condition. Then there exists a closed $2\pi i$-periodic subset $X \subset J(F)$ with the following properties:*

(1) $F(X) \subset X$;
(2) *each connected component C of X is a closed arc to infinity;*
(3) *there exists $K' > 0$ with the following property: If $z \in J(F)$ such that $\operatorname{Re} F^j(z) \geq K'$ for all $j \geq 0$, then $z \in X$.*

If F is of disjoint type, then we may choose $X = J(F)$.

The goal of this paper is to show that X can be chosen to be a Cantor bouquet.

5. Dynamical compactification

We now wish to compactify the Julia set of a disjoint-type entire function as in our theorem in such a way that the resulting set will turn out to be a one-sided hairy arc (where the original Julia set coincides precisely with the union of hairs).

Essentially, this is done as follows: It is easy to compactify the space of external addresses, by adding "intermediate addresses", such that the resulting space is homeomorphic to an arc. The desired space is then obtained by using this arc of addresses as the base of a comb, with the hair at address $\underline{s}$ attached at $\underline{s}$.

In fact, such a compactification of the dynamical plane can be defined completely generally for functions in the class $\mathcal{B}$ (or $\mathcal{B}_{\log}^{\mathrm{p}}$), without assuming that the escaping set is a union of hairs, or that the function is hyperbolic. This type of compactification has already been used extensively in the exponential family; compare e.g. [21, Section 2] and [25, Section 2], although it does not seem clear whether the precise details have been published previously. It is likely that this construction will be useful in future applications, and we hence use this opportunity to give a detailed account in the general setting.

The construction consists of two steps: Firstly, we must compactify the space of addresses $\mathcal{A}^{\mathbb{N}_0}$ for a given map $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ to the desired arc $\tilde{\mathcal{S}}$. Then we define a topology on $\tilde{H} := \overline{H} \cup \tilde{\mathcal{S}}$ with the desired properties.

Finally, we shortly discuss the corresponding results in the original dynamical plane of an entire function $f \in \mathcal{B}$ (for future reference).

We begin with the construction of the compactification $\tilde{\mathcal{S}}$.

Theorem 5.1. *There exists a totally ordered set $\tilde{\mathcal{S}} \supset \mathcal{A}^{\mathbb{N}_0}$ (where the order on $\tilde{\mathcal{S}}$ agrees with lexicographic order on $\mathcal{A}^{\mathbb{N}_0}$) with the following properties:*

(1) *With the order topology, $\tilde{\mathcal{S}}$ is homeomorphic to a line segment. In particular, $\tilde{\mathcal{S}}$ is compact and order-complete (i.e., every nonempty subset that is bounded from above has a supremum).*

(2) $\mathcal{A}^{\mathbb{N}_0}$ *is dense in $\tilde{\mathcal{S}}$.*

(Furthermore, this compactification is unique; i.e. any other ordered set with these properties is order-isomorphic to $\tilde{\mathcal{S}}$, with the order-isomorphism restricting to the identity on $\mathcal{A}^{\mathbb{N}_0}$.)

This result is actually (a special case of) a standard fact from the theory of ordered spaces. However, it is possible—and useful in the following—to give an explicit description of the space $\tilde{\mathcal{S}}$. We comment below on the particularly simple case where the number of tracts is finite up to translations by multiples of $2\pi i$, as is the case for functions of finite order.

To begin, let $F : \mathcal{T} \to H$ be a function in the class $\mathcal{B}_{\log}^{\mathrm{p}}$. Recall that the symbolic alphabet $\mathcal{A}$ is the set of tracts of F (that is, the set of components T of $\mathcal{T}$), equipped with a natural total order. Note also that $\mathcal{A}$ is countable.

Now we form an extended (also totally ordered) alphabet $\overline{\mathcal{A}}$ by adding "intermediate" entries, corresponding to "Dedekind cuts" of the set $\mathcal{A}$.

Definition 5.2 (Intermediate entries). An *intermediate entry* is a pair (A^-, A^+) with $\mathcal{A} = A^- \cup A^+$, $A^-, A^+ \neq \emptyset$ and $a^- < a^+$ for all $a^- \in A^-$ and $a^+ \in A^+$.

We define $\overline{\mathcal{A}}$ to be the union of $\mathcal{A}$ and the set of all intermediate entries.

Lemma 5.3 (Total order on $\overline{\mathcal{A}}$). *There is a natural total order on $\overline{\mathcal{A}}$ that agrees with the vertical order on $\mathcal{A}$, and $\overline{\mathcal{A}}$ is order-complete.*

Note that, when there are only finitely many tracts (up to $2\pi i\mathbb{Z}$-translations), this construction simply adds an intermediate entry between any pair of adjacent tracts. Otherwise, there will also be intermediate entries corresponding to "limit points" of sequences of tracts.

It may be worth pointing out that the construction differs from the standard construction of the real numbers from the rationals using Dedekind cuts, in that every element $a \in \mathcal{A}$ will be isolated in the space $\overline{\mathcal{A}}$. More precisely, there are two intermediate entries, above and below a, which separate a from all other elements of $\mathcal{A}$.

Nonetheless, the proof of order-completeness remains essentially the same: Let $B \subset \overline{\mathcal{A}}$ be bounded and nonempty; we must show the existence of $\sup B$. If B has a maximal element, then we are done. Otherwise, set

$$A^+ := \{a \in \mathcal{A} : b < a \text{ for all } b \in B\} \quad \text{and} \quad A^- := \mathcal{A} \setminus A^+.$$

Then it is easy to check that $(A^-, A^+) = \sup B$.

We can now describe the compactification $\tilde{\mathcal{S}}$ of the space of addresses as follows.

Definition 5.4. A (finite) sequence $\underline{s} = T_0 T_1 \ldots T_{n-1} S_n$, where $n \geq 0$, $T_j \in \mathcal{A}$ for $0 \leq j \leq n-1$, and where $S_n \in \overline{\mathcal{A}} \setminus \mathcal{A}$ is an intermediate entry, is called an *intermediate external address*.

We define $\mathcal{S}$ to be the union of the set $\mathcal{A}^{\mathbb{N}_0}$ of all external addresses and the set of all intermediate external addresses.

Lemma 5.5. *Then (with respect to lexicographic order), the space $\mathcal{S}$ is order-isomorphic to $\mathbb{R}$.*

In particular, the space $\tilde{\mathcal{S}} := \mathcal{S} \cup \{-\infty, +\infty\}$ is compact when equipped with the order topology (where $-\infty < \underline{s} < +\infty$ for all $\underline{s} \in \mathcal{S}$), and homeomorphic to $[0,1]$.

Proof. It is not difficult to see that the space $\tilde{\mathcal{S}}$ is compact, metrizable and connected, and hence homeomorphic and order-isomorphic to $[0,1]$ by a general theorem of topology [19, Theorems 6.16 and 6.17]. Instead, we sketch a simple direct proof with a dynamical flavor.

Because $\mathcal{A}$ is countable, and by definition of $\overline{\mathcal{A}}$, we can partition the real line $\mathbb{R}$ into sets $I(a) \subset \mathbb{R}$, $a \in \overline{\mathcal{A}}$ such that:

- For each $a \in \mathcal{A}$, the set $I(a)$ is an open interval.
- For each $a \in \overline{\mathcal{A}} \setminus \mathcal{A}$, the set $I(a)$ is a singleton.
- The sets $I(a)$ are pairwise disjoint and their order on the real line coincides with the order on $\overline{\mathcal{A}}$. That is, if $a, b \in \overline{\mathcal{A}}$ and $a < b$, then $x < y$ for all $x \in I(a)$ and $y \in I(b)$.

- $\bigcup_{a \in \overline{\mathcal{A}}} I(a) = \mathbb{R}$.

(This can be done by identifying $\mathcal{A}$ with a countable discrete—but not necessarily closed—subset of $\mathbb{R}$, and choosing an interval $I(a)$ around each $a \in \mathcal{A}$ such that the complement of the union of these intervals consists of singletons. We leave the details to the reader.)

Now define a continuous function

$$h : \bigcup_{a \in \mathcal{A}} I(a) \to \mathbb{R}$$

such that $h'(x) \geq 2$ everywhere and such that $h(I(a)) = \mathbb{R}$ for each $a \in \mathcal{A}$ (so $h|_{I(a)} : I(a) \to \mathbb{R}$ is a diffeomorphism). Then the desired homeomorphism between $\mathbb{R}$ and the space $\mathcal{S}$ is provided by the symbolic dynamics of the map h. More precisely, for $x \in \mathbb{R}$, let $\psi(x)$ be the (unique) finite or infinite sequence $\underline{s} = s_0 s_1 \dots$ satisfying $h^j(x) \in I(s_j)$ (whenever defined). It is easy to check that this map is indeed an order-isomorphism between $\mathbb{R}$ and $\mathcal{S}$. $\qquad\square$

It now remains to define a topology on the set $\tilde{H} := \overline{H} \cup \tilde{\mathcal{S}}$. This topology will agree with the induced topology on $\overline{H}$, so we only need to specify a neighborhood base for every $\underline{s} \in \tilde{\mathcal{S}}$. This can be done as follows. Denote by H_R the right half plane $H_R = \{a + ib : a > R, b \in \mathbb{R}\}$; we will always assume R to be chosen sufficiently large that $H_R \subset H$.

(1) Let $\underline{s} = T_0 T_1 T_2 \cdots \in \mathcal{A}^{\mathbb{N}_0}$ be an infinite external address, and let $n \geq 0$. Let $R > 0$, and let V be the unbounded component of $T_n \cap H_R$. If R was chosen sufficiently large, then there is a branch ψ of F^{-n} defined on H_R such that $U := \psi(V)$ satisfies $F^j(U) \subset T_j$ for $0 \leq j \leq n$. For all such n and R, the set

$$\tilde{U} := U \cup \{\underline{t} \in \tilde{\mathcal{S}} : \text{the first } n \text{ entries of } \underline{t} \text{ are } T_0 T_1 \dots T_n\}$$

is a neighborhood of $\underline{s}$ by definition.

(2) Let $\underline{s} = T_0 T_1 \dots T_{n-1} S_n$ $(n \geq 0)$ be an intermediate external address. Let $T_n^+, T_n^- \in \mathcal{A}$ with $T_n^- < S_n < T_n^+$, and let $T_{n+1}^+, T_{n+1}^- \in \mathcal{A}$ be arbitrary. Also let R be sufficiently large; then $H_R \setminus (\overline{F_{T_n^-}^{-1}(T_{n+1}^-) \cup F_{T_n^+}^{-1}(T_{n+1}^+)})$ has a unique unbounded connected component V that lies between $F_{T_n^-}^{-1}(T_{n+1}^-)$ and $F_{T_n^+}^{-1}(T_{n+1}^+)$. Choosing R larger, if necessary, we set $U := \psi(V)$, where ψ is a branch of F^{-n} chosen such that $F^j(U) \subset T_j$ for $0 \leq j \leq n - 1$.

For every such T_n^-, T_n^+ and R, the set

$$\tilde{U} := U \cup \{\underline{t} \in \tilde{\mathcal{S}} : T_0 \dots T_{n-1} T_n^- T_{n+1}^- < \underline{t} < T_0 \dots T_{n-1} T_n^+ T_{n+1}^+ \text{ (lexicographically)}\}$$

is a neighborhood of $\underline{s}$ by definition.

(3) Let $\underline{s} = +\infty$ (the definition for $-\infty$ is analogous), let $T \in \mathcal{A}$ and let Γ be a Jordan arc that connects ∂T to ∂H. Let U be the connected component of $\overline{H} \setminus (\overline{T} \cup \Gamma)$ that contains points "above" T (i.e., that contains a sequence $a + ib_n$, with a fixed and $b_n \to +\infty$).

Then, for all T and Γ as above, the set

$$\tilde{U} := U \cup \{+\infty\} \cup \{\underline{t} \in \tilde{\mathcal{S}} : \text{the first entry of } \underline{t} \text{ is larger than } T\}$$

is a neighborhood of $+\infty$ by definition.

It is not difficult to check that each of the neighborhoods described above is in fact an open subset of $\tilde{H}$.

Proposition 5.6 (Topology of $\tilde{H}$)**.** *Equipped with the topology described above, the space $\tilde{H}$ is homeomorphic to the closed unit disk.*

Sketch of proof. By Urysohn's metrizability theorem [15, page 125, Theorem 16], we see that $\tilde{H}$ is metrizable. Furthermore, the space is compact. (We only need to show that every sequence $z_n \in \overline{H}$ with $z_n \to \infty$ in $\mathbb{C}$ has a convergent subsequence in $\tilde{H}$; this is not difficult to verify.) Since the neighborhoods given are connected, the space is locally connected.

Furthermore, let us denote by $\partial \tilde{H}$ the set $\partial H \cup \tilde{\mathcal{S}}$, which is topologically a circle. Suppose γ is a "crosscut", that is, an arc that connects two points of $\partial \tilde{H}$, with all points of γ except for the endpoints belonging to H. Then one can verify that $\tilde{H} \setminus \gamma$ has exactly two components, each of which intersects the boundary circle $\partial \tilde{H}$ in an arc. From this, it follows that $\tilde{H}$ is homeomorphic to the closed unit disk. (E.g. glue a disk inside the boundary circle, and use the Kline sphere characterization [5] to show that the resulting space is homeomorphic to the 2-sphere.) $\square$

Let T be a tract of F, and let $\tilde{T}$ be the closure of T in $\tilde{H}$. Then it follows from the definitions that the homeomorphism $F : \overline{T} \to \overline{H}$ extends to a homeomorphism $F : \tilde{T} \to \tilde{H}$.

Properties. The topology we introduced—and particularly its analog in the original dynamical plane—has a number of further useful properties, again similarly to the setting of exponential maps. For example, any dynamically natural curve (e.g. one which does not contain escaping points) that tends to infinity will have a unique endpoint on the boundary circle. However, we shall restrict to those facts that are relevant to our paper.

Proposition 5.7. *Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, and let $\underline{s}$ be an external address. If $z_n \in J_{\underline{s}}(F)$ is a sequence with $\operatorname{Re} z_n \to \infty$, then $z_n \to \underline{s}$ in the topology of $\tilde{H}$.*

In particular, $J_{\underline{s}}(F) \cup \{\underline{s}\}$ is a compact subset of $\tilde{H}$. If F is of disjoint type, then $J_{\underline{s}}(F) \cup \{\underline{s}\}$ is connected.

Proof. It follows from the definition of the topology on $\tilde{H}$ that points in $J_{\underline{s}}(F)$ cannot accumulate on any element of $\tilde{\mathcal{S}}$ apart from $\underline{s}$, which implies the first two claims. The fact that $J_{\underline{s}}(F) \cup \{\underline{s}\}$ is connected when F is of disjoint type follows from the fact that $J_{\underline{s}}(F)$ is the countable intersection of closed, unbounded, connected sets by definition, and hence every connected component of $J_{\underline{s}}(F)$ is unbounded. (In fact it follows from [22] that $J_{\underline{s}}(F)$ is connected as a subset of $\mathbb{C}$, but we shall not require this.) $\square$

Proposition 5.8 (Hairy subsets of $J(F)$)**.** *Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, and let $X \subset J(F)$ be a closed set such that*

(1) *for every $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$, the set $X_{\underline{s}} := J_{\underline{s}}(F) \cap X$ is a closed arc to infinity;*
(2) *the set of addresses $\underline{s} \in \mathcal{A}^{\mathbb{N}_0}$ with $X_{\underline{s}} \neq \emptyset$ is dense in $\mathcal{A}^{\mathbb{N}_0}$.*

(If F satisfies a uniform head-start condition, then these properties are satisfied by the set X from Corollary 4.4.)

Let us denote by $\tilde{X}$ the closure of X in the space $\tilde{H}$. Then $\tilde{X} = X \cup \tilde{S}$ is a continuum and, with $B := \tilde{S}$, we have:

(1) *The closure of every component of $\tilde{X} \setminus B = X$ is an arc, with exactly one endpoint in B;*

(2) *distinct components of X have disjoint closures in $\tilde{X}$;*

(3) *the set X is dense in $\tilde{X}$.*

Proof. $\tilde{H}$ is a compact metric space, and hence so is $\tilde{X}$. By the second assumption on X, we have $\tilde{S} \subset \tilde{X}$. Hence $\tilde{X} = X \cup \tilde{S}$ is connected, and can be written as the disjoint union of the arc B and the (half-open) rays $X_{\underline{s}} := J_{\underline{s}}(F) \cap X$, with each ray ending at the point $\underline{s} \in B$ by the previous Proposition. This establishes the first two claims, and the third is satisfied by definition.

Finally, suppose that F satisfies a uniform head-start condition and let X be the set from Corollary 4.4 and let $R > 0$. By Lemma 3.4 (applied to the restriction of F to $F^{-1}(H_R)$), for every periodic address $\underline{s}$ there is a point $z \in J(F)$ with $\mathrm{addr}(z) = \underline{s}$ and $\mathrm{Re}\, F^j(z) \geq R$ for all j. This shows that, in this setting, the set X does indeed satisfy our hypotheses. $\square$

The case of finitely many tracts. After the preceding, somewhat abstract, discussion, let us return to the case where F is the logarithmic transform of an entire function $f \in \mathcal{B}$ which has only finitely many, say N, tracts over infinity. As discussed above, $\mathcal{A}$ can be identified with the set of pairs (s, j), where $j \in \{1, \ldots, N\}$ and $s \in \mathbb{Z}$, where $(s_1, j_1) > (s_2, j_2)$ if $s_1 > s_2$ or $s_1 = s_2$ and $j_1 > j_2$.

Now, extending $\mathcal{A}$ to the set $\overline{\mathcal{A}}$ involves simply adding an "intermediate entry" between any two adjacent tracts, so we can identify $\overline{\mathcal{A}}$ with the set

$$\left\{ (s, j) : j \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, \ldots, N \right\}, s \in \mathbb{Z} \right\}.$$

The original dynamical plane. Suppose that the map $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ is a logarithmic transform of a function $f \in \mathcal{B}$. Then the translation $z \mapsto z + 2\pi i$ extends to a homeomorphism of $\tilde{H}$ to itself. We can quotient $\tilde{H} \setminus \{-\infty, +\infty\}$ to obtain a compactification of the original dynamical plane of f, with a "circle of addresses" at infinity. Again, this space will be homeomorphic to a closed unit disk, with the open disk corresponding to the original dynamical plane.

6. Proof of Theorem 1.5

With Propositions 3.5 and 5.8, we have established most of the conditions required to show that $J(F)$ is a Cantor bouquet, and hence establish Theorem 1.5. It only remains to show that, when $z_0 \in J(F)$ and $z_n \in J(F)$ with $z_n \to z_0$, then the arcs connecting z_n to infinity converge to the corresponding arc for z_0 in the Hausdorff metric (provided F satisfies a uniform head-start condition).

The main fact needed is as follows, where we use the speed ordering $\prec$ as defined in Proposition 4.3.

Proposition 6.1. *Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ be a function that satisfies a uniform head-start condition. Suppose that $a_n \in J(F)$ converges to a point $a \in J(F)$, and that, for each n,*

$b_n \in J_{\mathrm{addr}(a_n)}(F)$ *has the same external address as* a_n *and satisfies* $a_n \preceq b_n$ *in the speed ordering of* F.

If $b \in J(F)$ *is an accumulation point of the sequence* b_n, *then then* $a \preceq b$.

Proof. We prove the contrapositive: If $a \succ b$, then $a_n \succ b_n$ for sufficiently large n.

Indeed, by definition of the speed ordering, there exists $k \geq 0$ such that $\operatorname{Re} F^k(a) > \varphi(\operatorname{Re} F^k(b))$, where φ is the function from the head-start condition. Hence, for sufficiently large n, we also have $\operatorname{Re} F^k(a_n) > \varphi(\operatorname{Re} F^k(b_n))$, by continuity, and thus $a_n \succ b_n$, as claimed. $\square$

Proposition 6.2. *Let* $F \in \mathcal{B}^{\mathrm{p}}_{\log}$ *satisfy a uniform head-start condition, let* X *be as in Corollary 4.4 and denote the closure of* X *in* $\tilde{H}$ *by* $\tilde{X}$.

Then $\tilde{X}$ *is a comb.*

Proof. For any $z \in \tilde{X}$, let us denote by A_z the unique arc connecting z to $\tilde{S}$ in $\tilde{X}$. Then

$$A_z = \{z\} \cup \{w \in J_{\underline{s}}(F) : w \succ z\} \cup \{\underline{s}\},$$

where $\underline{s} = \mathrm{addr}(z)$. By Proposition 5.8, it only remains to check the requirement that, if a sequence $z_n \in \tilde{X}$ converges to a point $z_0 \in \tilde{X}$, then the arcs A_{z_n} converge to A_{z_0} in the Hausdorff metric.

Passing to a subsequence, we may assume that A_{z_n} is a convergent sequence in the Hausdorff metric; let the limit be A', say. Clearly A' must be a subset of the arc $X_{\underline{s}} \cup \{\underline{s}\}$, where $\underline{s} = \mathrm{addr}(z_0)$. Since A' is connected (as the Hausdorff limit of compact connected subsets of the compact space $\tilde{X}$) and contains both z_0 and $\underline{s}$, we must have $A_{z_0} \subset A'$.

On the other hand, Proposition 6.1 shows that $A' \subset A_{z_0}$, and hence $A_{z_0} = A'$. This proves that $\tilde{X}$ is a comb. $\square$

Corollary 6.3 (Julia set is a hairy arc). *Let* $F \in \mathcal{B}^{\mathrm{p}}_{\log}$ *be of disjoint type and satisfy a uniform head-start condition. Let* $\tilde{J}(F)$ *denote the closure of* $J(F)$ *in the topology of* $\tilde{H}$.

Then the set $\tilde{J}(F)$ *is a one-sided hairy arc. In particular, both* $J(F)$ *and* $\exp(J(F))$ *are Cantor bouquets.*

Proof. By Proposition 6.2, $\tilde{J}(F)$ is a comb, and Proposition 3.5 thus implies that $\tilde{J}(F)$ is a hairy arc, which is one-sided by construction. Using Theorem 2.4, this easily implies that $J(F)$ is a Cantor bouquet.

It is straightforward to see that $\exp(J(F))$ is also a Cantor bouquet. For example, we can use the above-mentioned extension of the exponential map to $\tilde{H} \setminus \{-\infty, +\infty\}$ to see that $J(f)$ is a brush of a "hairy circle" in the sense of [1] (which is the same has a hairy arc with the two endpoints of the base identified). $\square$

This completes the proof of Theorem 1.5.

7. The general case

To prove Theorem 1.6, it remains to consider the case of a function $F \in \mathcal{B}^{\mathrm{p}}_{\log}$ that is not necessarily of disjoint type. We have already seen that the set $X \subset J(F)$ from Corollary 4.4 can be compactified to a comb, but in general there is no reason to expect

that every hair in X has other hairs accumulating both from above and below. However, we shall show that X at least contains a Cantor bouquet. Let us define, for $R > 0$:

$$J^R(F) := \{z \in J(F) : \operatorname{Re} F^j(z) \geq R \text{ for all } j \geq 0\}.$$

Our goal now is to show:

Proposition 7.1 (Hairy arcs in Julia sets). *Let $F \in \mathcal{B}^{\mathrm{p}}_{\log}$ satisfy a uniform head-start condition, and let $\tilde{X}$ be the comb from Proposition 6.2. Then there exists a hairy arc $Z \subset \tilde{X}$ (again with base $B = \tilde{\mathcal{S}}$) such that $J^R(F) \subset Z$ for sufficiently large R, and such that $Z \setminus \tilde{\mathcal{S}}$ is $2\pi i$-periodic.*

In order to work with non-disjoint-type functions, we need the following well-known expansion estimate.

Lemma 7.2 ([12, Lemma 1]). *Let $F : \mathcal{T} \to H$ be an element of $\mathcal{B}^{\mathrm{p}}_{\log}$, and let K be sufficiently large that $H_K \subset H$. Then*

$$|F'(z)| \geq \frac{1}{4\pi}(\operatorname{Re} F(z) - \ln K) \quad \text{for all } z \in \mathcal{T}.$$

In particular, there is $R > 0$ such that $|F'(z)| \geq 2$ whenever $\operatorname{Re} F(z) \geq R$.

The following is the main dynamical fact that will allow us to construct a Cantor bouquet in $J(F)$.

Proposition 7.3 (Accumulation from above and below). *Let $F : \mathcal{T} \to H$ belong to the class $\mathcal{B}^{\mathrm{p}}_{\log}$, and let $\tau > 0$.*

Then there exists $\tau' \geq \tau$ with the following property. For every $z_0 \in J^{\tau'}(F)$, there exist sequences $z_n^-, z_n^+ \in J^\tau(F)$ with $\operatorname{addr}(z_n^-) < \operatorname{addr}(z_0) < \operatorname{addr}(z_n^+)$ for all n and $z_n^- \to z_0$, $z_n^+ \to z_0$.

Proof. Let R be the number from Lemma 7.2, so that $|F'(z)| \geq 2$ whenever $\operatorname{Re} F(z) \geq R$; also assume that R is large enough such that $H_R \subset H$. If we set $\tau' := \max(R, \tau) + \pi$, then the claim follows analogously to Proposition 3.5, using expansion in the Euclidean metric instead of expansion in the hyperbolic metric. $\square$

We are now ready to prove Proposition 7.1. For the remainder of the section, we fix the function $F \in \mathcal{B}^{\mathrm{p}}_{\log}$ (satisfying a uniform head-start condition) and the sets X and $\tilde{X}$. To begin, we essentially show that the comb $\tilde{X}$ can be "straightened", using the same proof as [1, Theorem 3.11].

Proposition 7.4 (Potential function). *There exists a continuous function $\rho : X \to [0, \infty)$ with the following properties.*

  (1) *ρ is $2\pi i$-periodic;*
  (2) *ρ is strictly increasing along hairs; i.e. if $z, w \in X$ have the same external address and $z \succ w$, then $\rho(z) > \rho(w)$;*
  (3) *For any sequence (z_n), we have $\rho(z_n) \to \infty$ if and only if $\operatorname{Re} z_n \to \infty$.*

Proof. For any point $x \in X$, let us denote by A_x the unique arc in $\tilde{X}$ that connects x to the base $B = \tilde{\mathcal{S}}$.

Recall that we can quotient the space $\tilde{X} \setminus \{-\infty, +\infty\}$ by the action of $z \mapsto z + 2\pi i$ (extended to $\tilde{\mathcal{S}}$ in the natural way) to obtain a compact, connected metric space Y. We denote the projection to the quotient by $\pi : \tilde{X} \setminus \{-\infty, +\infty\} \to Y$.

Let $\omega : Y \to [0, 1]$ be a *Whitney function*—also referred to as a *size function*—for Y. That is, ω is a continuous function on the set of non-empty compact subsets of Y (with the Hausdorff metric), taking values in $[0, 1]$, and having the following two properties. On the one hand, $\omega(\{x\}) = 0$ for all $x \in Y$, and on the other, ω is strictly increasing; i.e. if $A \subsetneq B$, then $\omega(A) < \omega(B)$. Such a function exists for any compact metric space (see e.g. [19, Exercise 4.33]).

We define
$$\rho : X \to [0, \infty), \quad \rho(x) := \frac{1}{\omega(\pi(A_x))}$$
and claim that ρ has the desired properties. Indeed, the function is $2\pi i$-periodic by definition, and the fact that ρ is increasing along hairs follows from the fact that ω is an increasing function. Finally, the "only if" direction in (3) follows from the first property of a Whitney function, while the "if" follows from the fact that $\tilde{X}$ is a comb. $\square$

Proof of Proposition 7.1. Let ρ be the function from the preceding Proposition, let τ be sufficiently large that $J^\tau(F) \subset X$ and choose τ' according to Proposition 7.3. Let K be such that $\operatorname{Re} z \geq \tau'$ whenever $\rho(z) \geq K$, and $R > 0$ such that $\rho(z) \geq K$ whenever $\operatorname{Re} z \geq R$.

We define
$$Z := \{z \in X : \rho(F^j(z)) \geq K \text{ for all } j \geq 0\} \cup \tilde{\mathcal{S}}.$$
Since ρ is a continuous function, it follows that Z is a compact subset of $\tilde{X}$. The fact that ρ is increasing along hairs implies that Z is a comb. The set $J^R(F)$ is contained in Z by definition. It remains to show that Z is a hairy arc.

So let $z_0 \in Z$. Let us assume first that z_0 is not an endpoint of Z, so that there is a point $w_0 \in Z$ with $\operatorname{addr}(w_0) = \operatorname{addr}(z_0)$ and $w_0 \prec z_0$. So there is some $M \geq 0$ such that $\operatorname{Re} F^M(z_0) > \varphi(\operatorname{Re} F^M(w_0))$, where φ is the function from the head-start condition.

Since $w_0 \in Z$, we have $w_0 \in J^{\tau'}(F)$. Thus we can apply Proposition 7.3: there exist sequences $w_n^-, w_n^+ \in J^\tau(F) \subset X$, converging to w_0, with $\operatorname{addr}(w_n^-) < \operatorname{addr}(w_0) < \operatorname{addr}(w_n^+)$. Since X is a comb, we can pick $z_n^\pm \succ w_n^\pm$ such that $z_n^- \to z_0$ and $z_n^+ \to z_0$. By continuity, we have $\operatorname{Re} F^M(z_n^\pm) > \varphi(\operatorname{Re} F^j(w_n^\pm))$ for sufficiently large n; without loss of generality, we may assume this holds for all n. Using the head-start condition, we have
$$\operatorname{Re} F^j(z_n^\pm) > \varphi(\operatorname{Re} F^j(w_n^\pm)) \geq \operatorname{Re} F^j(w_n^\pm)$$
for all $j \geq M$. Furthermore, let $\varepsilon > 0$ be such that $|z_n^\pm - w_n^\pm| \geq \varepsilon$ for all n. Since F is expanding, we have
$$|F^j(z_n^\pm) - F^j(w_n^\pm)| \geq 2^j \cdot \varepsilon.$$

Recall that the union $\mathcal{T}$ of tracts of F is $2\pi i$-periodic, that each tract is disjoint from its $2\pi i$-translates and that they accumulate only at ∞. Hence if $C > 0$ is sufficiently large, then the following holds: If $w \in \overline{\mathcal{T}}$ and $z \in \overline{\mathcal{T}}$ with $\operatorname{Re} z \geq \operatorname{Re} w$ and $|z - w| \geq C$, then $\operatorname{Re} w \geq R$.

So if $M_1 \geq M$ with $2^{M_1} \geq C/\varepsilon$, then
$$\operatorname{Re}(F^j(z_n^\pm) \geq R, \quad \text{and hence} \quad \rho(F^j(z_n^\pm)) \geq K$$

for all n and all $j \geq M_1$. Also, if n is large enough, then $\rho(F^j(z_n^{\pm})) > K$ for $j = 0, \ldots, M_1 - 1$ (by continuity). Thus $z_n^{\pm} \in Z$ for sufficiently large n, as desired.

If z_0 is an endpoint of Z, then we can pick a sequence of points on the hair of z_0. Each of these is not an endpoint and hence we can apply the fact we have just proved. By diagonalization, we can thus find sequences $z^-, z^+ \in Z$ with $z^{\pm} \to z_0$ and $\mathrm{addr}(z^-) < \mathrm{addr}(z_0) < \mathrm{addr}(z^+)$.

Thus we have verified that the comb Z is indeed a hairy arc. $\qquad\square$

As in the previous section, we obtain the following Corollary.

Corollary 7.5. *Let* $F \in \mathcal{B}_{\log}^p$. *Then the set X from Corollary 4.4 can be chosen such that X and $\exp(X)$ are Cantor bouquets.*

This completes the proof of Theorem 1.6.

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